ON AN ASYMPTOTIC METHOD APPLICABLE TO THE SOLUTION OF INTEGRAL EQUATIONS IN THE THEORY OF ELASTICITY AND IN MATHEMATICAL PHYSICS

(OB ODNOM ASIMPTOTICHESKOM METODE PRI RESHENII INTEGRAL'NYKH URAVNENII TEORII UPRUGOSTI I MATEMATICHESKOI FIZIKI)

PMM Vol.30, № 4, 1966, pp. 732-741

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(Received September 15, 1965)

The investigation deals with integral equations arising in certain mixed problems in plane elasticity, in particular plane contact problems concerned with the action of a rigid punch on an elastic layer located on a rigid substrate. The entire study is devoted to the construction of solutions to Equation (1.1) when the parameter a is large. Exact solutions to (1.1) are obtained in the form of series which hold for $a^* < a < \infty$. A corollary to the results obtained here provides a rigorous basis for the method given in [1]. Illustrative examples are given.

1. Consider the integral equation

$$\int_{-a}^{a} k(x-\xi)\, q_\eta(\xi)\, d\xi = \pi e^{i\eta x} \qquad (|x| \leqslant a,\ \operatorname{Im}\eta = 0) \tag{1.1}$$

whose kernel is of the form

$$k(t) = \int_0^\infty \frac{L(u)}{u} \cos tu\, du \tag{1.2}$$

Here, $L(u)/u$ is an even function which is real on the real axis and meromorphic on the complex plane.

The function $L(z)/z$ may be represented in the complex plane in the form

$$\frac{L(z)}{z} = \frac{P(z)}{Q(z)} = K(z) \tag{1.3}$$

where $P(z)$ and $Q(z)$ are entire functions with the asymptotic form of their zeros given by

$$z_n \sim \pm i(\beta n + b) \pm c_1 \ln n, \quad \zeta_n \sim \pm i(\beta n + g) \pm c_2 \ln n \qquad (n \to \infty) \tag{1.4}$$

All constants in (1.4) are real. It is assumed that the function $L(z)/z$ has the following properties:

$$\lim \frac{L(z)}{z} = A \quad \text{for } z \to 0, \quad \frac{L(z)}{z} = 0\,(z^{-2\gamma}) \quad \text{for } z \to \infty, \quad 0 < \gamma < 1 \tag{1.5}$$

Property (1.5) imposes the following interrelations among the constants β, b, g and γ

$$2(b-g) = \beta\gamma \qquad \text{for } c_1 \neq 0, \quad c_2 \neq 0$$
$$b - g = \beta\gamma \qquad \text{for } c_1 = 0, \quad c_2 = 0 \tag{1.6}$$

Assume that the zeros z_n and ζ_n are distinct. In the case of multiple roots or in case the asymptotic roots are somewhat different from (1.3), all of the results established below may be obtained in the same manner.

It is shown below that solution of Wiener-Hopf equation

$$\int_0^\infty k(x-\xi)\, f_\eta(\xi)\, d\xi = \pi e^{i\eta x} \qquad (0 \leqslant x \leqslant \infty) \tag{1.7}$$

results in the solution to an infinite system of linear algebraic equations (I)

Thereupon, having obtained the general form of the asymptotic solution of (1.1) for $a \to \infty$, we will show, that to obtain a solution to another infinite system of linear algebraic equations (II) which is perturbed about (I), we must transform it into (I). Accomplishing the transformation by the method of successive approximations, we obtain the exact solution to Equation (1.1) for $a^* < a < \infty$.

2. Let us note certain properties of the kernel $k(t)$ defined in (1.2).

Utilizing the properties of Fourier integrals, it may be shown that

$$k(t) = O(t^{2\gamma-1}), \qquad k(t) = O(\ln t) \tag{2.1}$$

respectively for $0 < \gamma < 0.5$ and $\gamma = 0.5$ for $t \to 0$. In case $0.5 < \gamma < 1$, the kernel $k(t)$ will be continuous. With the aid of the residue theory, the kernel $k(t)$ may be written in the form

$$k(t) = \sum_{r=1}^{\infty} s_r \exp i\zeta_r t \qquad (s_r = \pi i P(\zeta_k)\,[Q'(\zeta_k)]^{-1}) \tag{2.2}$$

with the series converging uniformly for all t, except for $t = 0$ with $0 < \gamma \leq 0.5$, in which case it has an integrable singularity.

Employing the Wiener-Hopf method, we will now obtain the solution to the integral equation (1.7) [2], which may be written as

$$f_\eta(t) = e^{i\eta t}[K(\eta)]^{-1} + \sum_{l=1}^{\infty} c_l(\eta) \exp iz_l t \qquad (|z_k| < |z_{k+1}|, \quad |\zeta_k| < |\zeta_{k+1}|) \tag{2.3}$$

Here

$$c_l(\eta) = [K_+(\eta)]^{-1}\,[(\eta - z_l)\,K_+'(-z_l)]^{-1} \tag{2.4}$$

$$K_+(\alpha) = \sqrt{A}\; \frac{\prod_{k=1}^{\infty} (1 + \alpha / z_k) \exp i\alpha / \beta k}{\prod_{k=1}^{\infty} (1 + \alpha / \zeta_k) \exp i\alpha / \beta k}$$

$$K_-(\alpha) = \sqrt{A}\,\frac{\prod\limits_{k=1}^{\infty}(1-\alpha/z_k)\exp(-\alpha i/\beta k)}{\prod\limits_{k=1}^{\infty}(1-\alpha/\zeta_k)\exp(-\alpha i/\beta k)} \tag{2.4}$$

with z_k and ζ_k located in the upper half-plane.

Note that (2.3) and (2.4) hold for the following values of η :

$$\operatorname{Im}\eta \geqslant -\operatorname{Im}\zeta_1, \qquad \eta \neq z_l$$

By making use of the known properties of Laplace transforms, it may be shown that

$$f_\eta(t) = O(t^{-\gamma}) \qquad (t \to 0) \tag{2.5}$$

The latter is established by investigating the asymptotic properties of the Laplace transform of the function $f_\eta(t)$ and making use of the estimate

$$K_+(\alpha) = O\left[\frac{\Gamma^k(1+g/\beta - i\alpha/\beta)}{\Gamma^k(1+b/\beta - i\alpha/\beta)}\right] \qquad (\alpha\to\infty,\ |{}^1/_2\pi - \arg\alpha| < \pi) \tag{2.6}$$

with $k = 2$ in the first case of (1.6) and $k = 1$ in the second case; $\Gamma(x)$ is Euler's gamma function.

We now rewrite integral equation (1.7) employing the form of the kernel $k(x-\xi)$ given by (2.2) and the form of the solution $f_\eta(\xi)$ given by (2.3), and integrate the result, bringing the integral sign inside the double summation sign. The justification for this procedure lies in the fact that there is only a finite number of points at which the series is not uniformly convergent, and at these points the function is integrable.

The result thus obtained is

$$\int_0^\infty k(x-\xi) f_\eta(\xi)\,d\xi = \frac{2i}{K(\eta)}\sum_{r=1}^\infty \frac{s_r\zeta_r}{\zeta_r^2-\eta^2}\exp i\eta x +$$
$$+ 2i\sum_{l=1}^\infty c_l(\eta)\left(\sum_{r=1}^\infty \frac{s_r\zeta_r}{\zeta_r^2 - z_l^2}\right)\exp iz_l x -$$
$$- i\sum_{r=1}^\infty s_r\left(\frac{1}{K(\eta)}\frac{1}{\zeta_r-\eta} + \sum_{l=1}^\infty \frac{c_l(\eta)}{\zeta_r - z_l}\right)\exp i\zeta_r x \tag{2.7}$$

The summation of the series in the first term in (2.7) may be found in the following manner:

$$\frac{2i}{K(\eta)}\sum_{r=1}^\infty \frac{s_r\zeta_r}{\zeta_r^2-\eta^2} = \frac{1}{K(\eta)}\lim_{x\to\infty}\int_0^\infty k(x-\xi)e^{i\eta(\xi-x)}\,d\xi = \tag{2.8}$$
$$= \frac{1}{K(\eta)}\lim_{x\to\infty}\left[\int_{-\infty}^{\infty} k(y)e^{i\eta y}\,dy + \int_{-\infty}^{-x} k(y)e^{i\eta y}\,dy\right] = \frac{1}{K(\eta)}\int_{-\infty}^{\infty} k(y)e^{i\eta y}\,dy = \pi$$

Here the asymptotic behavior of the zeros of (1.4) has been taken into account.

The right-hand side of (2.7) must be equal to $\pi e^{i\eta x}$, inasmuch as $f_\eta(x)$ is an exact solution of integral equation (1.7). Hence, taking into account (2.8), we have the identity

$$2\sum_{l=1}^{\infty} c_l(\eta)\left(\sum_{r=1}^{\infty}\frac{s_r\zeta_r}{\zeta_r^2-z_l^2}\right)\exp iz_l x -$$
$$-\sum_{r=1}^{\infty} s_r\left(\frac{1}{K(\eta)}\frac{1}{\zeta_r-\eta}+\sum_{l=1}^{\infty}\frac{c_l(\eta)}{\zeta_r-z_l}\right)\exp i\zeta_r x\equiv 0 \qquad (2.9)$$

In view of the linear independence of the functions $\exp iz_l x$ and $\exp i\zeta_r x$, the immediately preceding identity yields directly

$$\sum_{r=1}^{\infty}\frac{s_r\zeta_r}{\zeta_r^2-z_l^2}=0 \qquad (l=1,2,\ldots) \qquad (2.10)$$

$$(1)\quad \frac{1}{K(\eta)}\frac{1}{\zeta_r-\eta}+\sum_{l=1}^{\infty}\frac{c_l(\eta)}{\zeta_r-z_l}=0 \qquad (r=1,2,\ldots) \qquad (2.11)$$

Thus, if $f_\eta(x)$ is a solution of integral equation (1.7), the coefficients $c_l(\eta)$ satisfy the infinite system of equations (2.11), together with (2.10).

We will now determine the general form of the solution $q_\eta(x)$ to the integral equation (1.1). Namely, we will show that it takes the form

$$q_\eta(x)=B_0(a,\eta)e^{i\eta x}+\sum_{k=1}^{\infty}[B_k^+(a,\eta)\exp iz_k(a+x)+$$
$$+B_k^-(a,\eta)\exp iz_k(a-x)] \qquad (2.12)$$

As a preliminary step, we will examine the result of substituting the zeroth term in the asymptotic form [1] into integral equation (1.1)

$$q_\eta^\circ(x)=e^{-i\eta a}f_\eta(a+x)+e^{i\eta a}f_{-\eta}(a-x)-K^{-1}(\eta)e^{i\eta x} \qquad (2.13)$$

The substitution of $q_\eta^\circ(x)$ into (1.1) and integration, taking into account (2.8), (2.10) and (2.11), yields

$$\int_{-a}^{a}k(x-\xi)q_\eta^\circ(\xi)d\xi=$$
$$=\pi e^{i\eta x}+i\sum_{r=1}^{\infty}s_r\sum_{l=1}^{\infty}\frac{c_l(\eta)}{\zeta_r+z_l}\exp[2aiz_l+i\zeta_r(a-x)-i\eta a]+$$
$$+i\sum_{r=1}^{\infty}s_r\sum_{l=1}^{\infty}\frac{c_l(-\eta)}{\zeta_r+z_l}\exp[2aiz_l+i\zeta_r(a+x)+i\eta a] \qquad (2.14)$$

Let us examine the integral equation satisfied by the remainder of the terms in the asymptotic expansion entering (1.1). Writing the remainder of the terms in the form

$$q_\eta^*(x)=q_\eta(x)-q_\eta^\circ(x) \qquad (2.15)$$

(here $q_\eta(x)$ is the exact solution of (1.1)), we obtain, upon substitution into (1.1),

$$\int_{-a}^{a} k(x-\xi)\, q_\eta^*(\xi)\, d\xi = -i \sum_{r=1}^{\infty} s_r \sum_{l=1}^{\infty} \frac{\exp 2aiz_l}{\zeta_r + z_l} \{c_l(\eta) \exp[i\zeta_r(a-x) - i\eta a] + c_l(-\eta) \exp[i\zeta_r(a+x) + i\eta a]\} \quad (2.16)$$

We now apply, in turn, to (2.16) the method of separating the zeroth term of the asymptotic expansion [1]. This may be accomplished by solving the integral equation

$$\int_{0}^{\infty} k(x-\xi)\, f_{\zeta_r}(\xi)\, d\xi = \pi \exp i\zeta_r x \qquad (0 \leqslant x < \infty) \quad (2.17)$$

Note that the zeroth term of the asymptotic expansion in the solution of (2.16), whose index will be one in the solution to Equation (1.1), will be found to contain a small parameter of the form $\exp 2aiz_l$.

The solution of the integral equation (2.17) may be obtained with the aid of (2.3) and (2.4), replacing η with ζ_r, whereupon the solution will be, as before, a combination of the functions $\exp iz_l t$. In the same manner, it may be found that all subsequent terms in the asymptotic solution $q_\eta(x)$ will have the same structure, since each step will require the solution of an equation of the form (2.17).

Thus, Formula (2.12) is proven.

It is now necessary to determine the coefficients $B_0(a, \eta)$, $B_k^+(a, \eta)$ and $B_k^-(a, \eta)$ in the expansion (2.12). We will show that these coefficients satisfy some infinite system of linear algebraic equations (II). For this purpose, we substitute $q_\eta(x)$, in the form given in (2.12) into integral equation (1.1), and take $k(t)$ in the form (2.2). Upon integration, and taking into account (2.8) and (2.10), we obtain

$$\int_{-a}^{a} k(x-\xi)\, q_\eta(\xi)\, d\xi = 2iB_0(a, \eta)\, e^{i\eta x} \sum_{r=1}^{\infty} \frac{s_r \zeta_r}{\zeta_r^2 - \eta^2} -$$

$$- i \sum_{r=1}^{\infty} s_r \left\{ \left[\frac{B_0(a, \eta)\, e^{-ia\eta}}{\zeta_r - \eta} + \sum_{l=1}^{\infty} \left(\frac{B_l^+(a, \eta)}{\zeta_r - z_l} + \frac{B_l^-(a, \eta) \exp 2aiz_l}{\zeta_r + \zeta_l} \right) \right] \exp i\zeta_r(a+x) + \right.$$

$$\left. + \left[\frac{B_0(a, \eta)\, e^{ia\eta}}{\zeta_r + \eta} + \sum_{l=1}^{\infty} \left(\frac{B_l^-(a, \eta)}{\zeta_r - z_l} + \frac{B_l^+(a, \eta) \exp 2aiz_l}{\zeta_r + z_l} \right) \right] \exp j\zeta_r(a-x) \right\} \quad (2.18)$$

We now require that the right-hand side of (2.18) equals $\pi e^{i\eta x}$. This leads to the following conditions

$$B_0(a, \eta) = K^{-1}(\eta)$$

$$\frac{B_0 e^{-i\eta a}}{\zeta_r - \eta} + \sum_{l=1}^{\infty} \left(\frac{B_l^+(a, \eta)}{\zeta_r - z_l} + \frac{B_l^-(a, \eta) \exp 2aiz_l}{\zeta_r + z_l} \right) = 0$$

$$\frac{B_0 e^{i\eta a}}{\zeta_r + \eta} + \sum_{l=1}^{\infty} \left(\frac{B_l^-(a, \eta)}{\zeta_r - z_l} + \frac{B_l^+(a, \eta) \exp 2aiz_l}{\zeta_r + z_l} \right) = 0 \qquad (r = 1, 2, \ldots) \quad (2.19)$$

If we can find the $B_l^+(a, \eta)$ and $B_l^-(a, \eta)$, which satisfy the infinite system (2.19), we will have obtained the solution to Equation (1.1).

Comparing the second and third systems of equations in (2.19), we obtain

$$B_l^+(a, \eta) = B_l^-(a, -\eta) \tag{2.20}$$

Define

$$x_l^{\mp} = B_l^+(a, \eta) \pm B_l^-(a, \eta) \tag{2.21}$$

Addition and subtraction of the second and third systems of equations in (2.19) leads to the conclusion that, in order to find the solution of Equation (1.1), it is sufficient to obtain the solution of an infinite system of equations of the form (case x_l^+)

$$\text{(II)} \sum_{l=1}^{\infty}\left(\frac{1}{\zeta_r - z_l} + \frac{\exp 2aiz_l}{\zeta_r + z_l}\right) x_l + \frac{e^{-i\eta a}K^{-1}(\eta)}{\zeta_r - \eta} + \frac{e^{+i\eta a}K^{-1}(\eta)}{\zeta_r + \eta} = 0 \quad (r = 1, 2, \ldots) \tag{2.22}$$

Hereinafter, in addition to (2.11), it will be necessary to consider a system of the form (2.11) but with $\eta = \zeta_r$. Constructing the above system of equations similarly to (2.11), we obtain

$$\frac{1}{\pi i}\sum_{l=1}^{\infty}\frac{s_r c_l(\zeta_r)}{\zeta_k - z_l} = \begin{cases} 0 & \text{for } k \neq r, \quad r, k = 1, 2, \ldots \\ 1 & \text{for } k = r \end{cases} \quad \left(\tau_{lr} = \frac{s_r c_l(\zeta_r)}{\pi i}\right) \tag{2.23}$$

The problem may now be stated as follows: given the solutions to systems (2.11) and (2.23), find the x_l in system (2.22).

3. Let us write the problem in matrix form. Introduce the notation

$$a_{r,l} = \frac{1}{\zeta_r - z_l}, \qquad b_{r,l} = \frac{1}{\zeta_r + z_l}\exp 2aiz_l, \qquad d_r = \frac{2(\zeta_r \cos\eta a - \eta \sin \eta a)}{K(\eta)(\eta^2 - \zeta_r^2)}$$

$$x^\circ_l(\eta) = c(\eta)e^{-i\eta a} + c_l(-\eta)e^{+i\eta a}, \qquad \tau_{l,r} = \frac{-1}{K_+(\zeta_r)K_+'(-z_l)(z_l - \zeta_r)[K^{-1}(\zeta_r)]'} \tag{3.1}$$

By utilizing (1.4), (2.4) and (2.6), together with the properties of Euler's gamma functions, the following estimates are obtained:

$$c_k(\eta)|_{k\to\infty} = O(k^{\gamma-1}), \qquad |\tau_{k,r}|_{k\to\infty} = O(k^{\gamma-1}), \qquad |\tau_{k,r}|_{r\to\infty} = O(r^{1-\gamma}) \tag{3.2}$$

The second estimate in (3.2) is for fixed r; the third estimate is for fixed k. We now introduce the matrices

$$A = (a_{r,l}), \qquad B(a) = (b_{r,l}), \qquad {}^{-1}A = (\tau_{l,r}), \qquad D = (dr) \tag{3.3}$$

as well as the column matrices $X = (x_l)$ and $X_0 = (x_l^\circ(\eta))$

The problem formulated at the end of Section 2 may now be stated as follows.

To find a column matrix X_0 satisfying the equation

$$(A + B(a))X = D \tag{3.4}$$

if we are given the column matrix X_0 satisfying

$$AX_0 = D \tag{3.5}$$

as well as ^{-1}A , the right-hand inverse of A , i.e.

$$A \cdot {}^{-1}A = I \tag{3.6}$$

where I is the unit matrix.

Note that the matrix $B(a)$ obviously has the following property

$$B(a) \to 0 \quad (a \to \infty) \quad (0 \text{ is the zero matrix}) \tag{3.7}$$

Reformulate (3.4) in terms of Y , which is defined by

$$X = X_0 + Y \tag{3.8}$$

so that we obtain, with the aid of (3.5),

$$AY = -B(a)Y - B(a)X_0 \tag{3.9}$$

We seek a solution in the form

$$Y = {}^{-1}AZ \tag{3.10}$$

(Z is a new unknown), and utilize (3.6) to obtain

$$Z = -B(a) \cdot {}^{-1}AZ - B(a)X_0 \tag{3.11}$$

Note that, in order to make use of the right inverse employed in obtaining (3.10), it is necessary to prove [3] the associative property of the product $A^{-1}AZ$. However, we can show that this operation is permissible without proving the associative property, by an investigation of the subsequent terms in the asymptotic series, obtained from Equation (2.16). Thus, Formulas (3.10) and (3.11) are proven.

Consider the matrix

$$U(a) = -B(a) \cdot {}^{-1}A \tag{3.12}$$

Its elements have the form (3.13)

$$u_{l,m} = -\sum_{k=1}^{\infty} b_{l,k}\tau_{k,m} = \frac{1}{K_+(\zeta_m)\,[K^{-1}(\zeta_m)]'} \sum_{k=1}^{\infty} \frac{\exp 2aiz_k}{(\zeta_l + z_k)\,K_+'(-z_k)(z_k - \zeta_m)}$$

From the last expression and the estimates in (3.2), it is clear that the matrix $U(a)$ exists for all $0 < a < \infty$ and

$$U(a) \to 0 \quad \text{for} \quad a \to \infty \tag{3.14}$$

The existence of the matrix $B(a)X_0$ may be shown in a similar manner.

Let us examine Equation (3.11) rewritten in the form

$$Z = U(a)Z - B(a)X_0 \tag{3.15}$$

in the space m of infinite, bounded sequences. The norm of the operator $U(a)$ is defined by

$$\| U(a) \| = \sup_{l} \sum_{m=1}^{\infty} |u_{l,m}| \tag{3.16}$$

We will show the correctness of this definition, and begin with the following estimates:

$$\| U(a) \| = \sup_l \sum_{m=1}^{\infty} \left| \sum_{k=1}^{\infty} b_{l,k} \tau_{k,m} \right| \leqslant \sup_l \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} | b_{l,k} \cdot \tau_{k,m} | \leqslant$$

$$\leqslant \sup_l \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} | b_{l,k} | \cdot | \tau_{k,m} | = \sup_l S_l(a) \qquad (3.17)$$

To prove the existence of $S_l(a)$ it is sufficient to establish the convergence of the iterated infinite series

$$\sum_{k=1}^{\infty} | b_{l,k} | \sum_{m=1}^{\infty} | \tau_{k,m} | \qquad (3.18)$$

for all l .

The inner series converges for every fixed k by virtue of (3.2). But then the iterated series also converges, since the coefficients $b_{l,k}$ decrease exponentially as $k \to \infty$.

Since series (3.18) converges absolutely, the corresponding doubly infinite series also converges, the summation of the two having the identical value.

It is evident from (3.1) that $|b_{1,k}| > |b_{l,k}|$ $(l = 2, 3, \ldots)$, so that

$$\sup_l S_l(a) = S_1(a) \qquad (3.19)$$

Thus, the existence of the norm (3.16) has been proven.

From the definition of the norm, it may be seen that

$$\| U(a) \| \to 0 \qquad \text{for } a \to \infty \qquad (3.20)$$

It follows from the above, that there exists an a_0 such that, for $a > a_0$ and $0 < q < 1$, the following inequality holds

$$\| U(a) \| < q \qquad (3.21)$$

But then it is easily shown that, for $a_0 < a < \infty$ the operator $U(a)$ maps from m into m . Since m is a Banach space, we can apply Banach's theorem on the existence of solutions to Equation (3.15) in the region $a_0 < a < \infty$ [4]. The solution itself may be obtained by the method of successive approximations, whereby convergence to a unique solution is obtained. Assuming the first term to be $B(a)X_0$, the solution to (3.15) is obtained in the form

$$Z = -\left(I + \sum_{k=1}^{\infty} U^k(a) \right) B(a) X_0 \qquad (3.22)$$

Or, using Formulas (3.8) and (3.10) to return to the original unknown X, we obtain the solution of Equation (3.4) in the form

$$X = X_0 - {}^{-1}A \left[I + \sum_{k=1}^{\infty} (-1)^k (B(a) \cdot {}^{-1}A)^k \right] B(a) X_0 \qquad (3.23)$$

In the same manner, the corresponding system yields x_l^-.

To determine the limit of applicability of Formula (3.23), it is necessary to solve Equation

$$\| U(a) \| = 1 \qquad (3.24)$$

the maximum positive root of which will be a_0. It is difficult to solve (3.24) in general form, but we may solve an approximate equation which yields an upper bound a^*, i.e. $a_0 < a^*$.

Such an equation is given by

$$S_1(a) = 1 \tag{3.25}$$

which may be represented in the form

$$\psi(a) \equiv \sum_{k=1}^{\infty} b_k e^{-2a\tau_k} - 1 = 0 \qquad \left(S = \sum_{k=1}^{\infty} b_k\right) \tag{3.26}$$

$$b_k = \frac{1}{|\zeta_1 + z_k|} \sum_{m=1}^{\infty} |\tau_{k,m}| > 0, \qquad \tau_k = \operatorname{Im} z_k > 0$$

Clearly, if $S < 1$, Equation (3.25) has no real roots and the solution (3.23) holds for $0 < a < \infty$.

If $S > 1$, Equation (3.26) has a single positive root a^*, and the solution (3.23) holds, at least, in the region $a^* < a < \infty$.

Since the curve (3.25) is convex downwards, Newton's method will always converge to a^*, provided the initial value of the root is taken as $0 < a_0^* < \infty$ such that $\psi(a_0^*) > 0$. If, for example, $b_r > 1$, then we can take

$$a_0^* = \frac{\ln b_r}{2\tau_r}$$

Actual computations show, that the a^* thus obtained is an upper bound for the limit of applicability. The reson for this lies in the fact that the method of successive approximations will converge if the following series converges

$$U(a) + U^2(a) + \ldots + U^n(a) + \ldots$$

Thus, we merely require that the foregoing series composed of the norm converge, whereas a^* is obtained from the approximate equation (3.25) which yields a larger root than Equation (3.24).

Examining the structure of the coefficients x_l in Formula (3.23), it may be seen that, indeed, for large a the zeroth term of the asymptotic solution is given by Formula (2.13), while the term whose index is one in the asymptotic solution is of order $O[\exp(-2a\tau_1)]$. The above explains the wide range of applicability of the zeroth term approximation (c.f. [1]).

4. As an example to which the above developed method may be applied, consider the case for which the function $u^{-1}L(u)$ is given by

$$u^{-1}L(u) = u^{-1}\tanh u \tag{4.1}$$

Aleksandrov, in investigating the mixed problem of plane torsion of an elastic layer [1], obtained a closed form solution of Equation (1.1) with the kernel as given in (4.1). He kindly communicated the result that, for $\eta = 0$, this solution is given by

$$q_0(x) = \frac{\pi e^{1/2\pi a}}{4K\left[e^{-\pi a}\right]\left(\sinh^2\frac{a\pi}{2} - \sinh^2\frac{x\pi}{2}\right)^{0.5}} = \frac{\displaystyle\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{(2k-1)!!\,(2r-1)!!}{2k!!\,2r!!}e^{\pi[(k+r)a+(r-k)x]}}{\displaystyle\sum_{m=0}^{\infty}\left[\frac{(2m-1)!!}{2m!!}\right]^2 e^{-2\pi a m}} \tag{4.2}$$

Here $K(x)$ is the complete elliptic integral.

We will now construct the solution for (1.1) by the method developed above.

For this purpose, we list certain necessary expressions

$$K(\alpha)=\frac{\tanh\alpha}{\alpha},\quad K_+(\alpha)=\frac{\Gamma(1/2-i\alpha/\pi)}{\Gamma(1-i\alpha/\pi)}\frac{1}{\sqrt{\pi}},\quad z_l=\pi li,\quad \zeta_r=(r-1/2)\pi i$$

$$a_{r,l}=\frac{1}{(r-l-1/2)\pi i},\quad b_{r,l}=\frac{\exp(-2a\pi l)}{(r+l-1/2)\pi i},\quad s_r=\frac{2}{2r-1}\qquad\left(\begin{matrix}l=1,2,\ldots\\ r=1,2,\ldots\end{matrix}\right)$$

$$c_l=\frac{(2l-1)!!}{(2l)!!},\quad \tau_{l,r}=\frac{(2r-3)!!\,(2l-1)!!}{i\,(2r-2)!!\,(2l-2)!!\,(2l-2r+1)}\tag{4.3}$$

and use Formulas (2.12) and (3.23) to construct the solution.

It is evident from Formulas (4.2), (2.12) and (3.23) that the zeroth terms of the asymptotic expansion of the two solutions are identical. Now compare the numerical coefficients of these terms in the asymptotic expansions whose index is one, for example the coefficient of $\exp^{-2\pi a-\pi(a+x)}$

In the solution obtained by Formula (3.23), this coefficient is produced by the matrix $^{-1}AB(a)X_0$, the general form of the element being

$$-\sum_{k=1}^{\infty}\sum_{r=1}^{\infty}\tau_{l,k}b_{k,r}x_r=\frac{1}{2}\frac{(2l-1)!!}{(2l-2)!!}\sum_{r=1}^{\infty}\left(\frac{(2r-1)!!}{2r!!}\right)^2\frac{\exp(-2\pi ar)}{(l+r)}\tag{4.4}$$

The necessary coefficient obtained from (4.4) when $r=l=1$ is $^1/_{16}$. In Formula (4.2), this coefficient consists of two parts: for $k=1$, $r=2$ and $m=0$ we get $^3/_{16}$, and for $k=0$, $r=1$, and $m=1$ we get $(-{}^1/_8)$. The sum of these two is $^1/_{16}$.

Similarly it will be found that the coefficient of $\exp[-2\pi a-2\pi(a+x)]$ equals $^1/_{16}$, and the coefficient of $\exp[-2\pi a-3\pi(a+x)]$ is $^{15}/_{256}$. In the same manner it can be shown that the coefficients of all other terms, as obtained by the two methods, are equal. In other words, Formulas (2.12) and (3.23) give the exact solution of integral equation (1.1) in case (4.1), for all $0<a<\infty$.

The proposed method may also be used for practical calculations, wherein the zeroth term of the asymptotic solution proves to be axtremely effective. In contact problems of elastic layers the zeroth term of the asymptotic solution generally completely covers the range for small and medium thicknesses and even extends partly into the region of large thicknesses.

As an example, consider integral equation (1.1) with $\eta=0$ for the case

$$\frac{L(u)}{u}=\tanh\frac{u}{2}\,\frac{u^4+3.526u^2+12.479}{u^4+2.522u^2+12.479}\,u^{-1}\tag{4.5}$$

This case approximates the kernel of the integral equation which arises in connection with the plane contact problem for an elastic strip which rests on a smooth rigid substrate and is acted on by a punch with a plane face [1]. The accuracy of approximation (4.5) is representing the function $L(u)/u$ for the contact problem is within 1.5% . Confining ourselves to the zeroth term of the asymptotic expansion in Formulas (2.12) and (3.23) and summing the respective series, we obtain an asymptotic solution in the form

$$q(x)=\frac{2\Delta}{h}\left[f(a+x)+f(a-x)-1\right]\tag{4.6}$$

$$f(t)=(1-\exp 2\pi t/h)^{-1/2}-0.113\exp(-1.627\,t/h)\sin(0.940\,t/h+0.436)+$$
$$+\,0.113\sin 0.436\exp(-3.13\pi t/h)\qquad\left(\Delta=\frac{E}{2(1-\sigma^2)},\quad \lambda=\frac{h}{a}\right)$$

The term $\exp(-3.13\pi t/h)$ in (4.6) arises as a result of the approximation in computing the series.

Below, we list some values of $aq(x)/\Delta$ as computed by means of Formula (4.6) for $\lambda = 2$ and various values of x/a. The third line lists, for comparison, the corresponding results obtained in [5] by a method appropriate for large λ.

$$\begin{array}{lcccccc} x/a = & 0, & 0.2, & 0.4, & 0.6, & 0.8, & 0.95 \\ aq(x)/\Delta = & 0.97, & 0.98, & 1.01, & 1.10, & 1.38, & 2.56 \\ aq(x)/\Delta = & 0.96, & 0.98, & 1.02, & 1.12, & 1.42, & 2.59 \end{array} \qquad (4.7)$$

The deviation in (4.7) does not exceed 3%.

In conclusion, let us note that the proposed method is also applicable to the solution of equations of the second kind

$$q(x) + \delta \int_{-a}^{a} k(x-\xi)\, q(\xi)\, d\xi = \pi\varphi(x) \qquad (4.8)$$

where, instead of $K(\alpha) = L(\alpha)/\alpha$, the function to be studied is $\pi^{-1} + \delta L(\alpha)/\alpha$.

The author is grateful to I.I. Vorovich and V.M. Aleksandrov for their constant interest in this work and their valuable advice.

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Translated by H.H.